1 Introduction

Flexible mechanical systems used to transport power, material and information are often modeled as axially moving beams. Tapes, band saws, power transmission chains and belts, paper sheets and threads, composite and textile fibers, and pipes transporting fluids are typical transport systems (Wickert and Mote, 1988; Paidoussis and Li, 1993). Further, dynamic analysis and control of axially moving beam vibration have received growing attention due to new applications of high speed magnetic tapes, flexible robotic manipulators, and flexible appendages on spacecrafts and structures (Tabarrok et al., 1974; Tsuchiya, 1983; Ibrahim and Mod, 1986; Yuh and Young, 1991; Tadikonda and Baruh, 1992).

A substantial body of literature related to passive or active vibration control of a stationary beam exists (Jiang, 1984; Bailey and Hubbard, 1985; Rahn and Mote, 1994). Little research, however, addresses control of vibration in a translating beam. Uncontrolled and truncated modes caused by the discretization of the continuous structure give rise to spillover in vibration control. One technique to overcome spillover is cancellation of disturbances by excitation of the structure or by modification of its reflection and transmission properties to dissipate vibration energy (von Flotow, 1986; Mace, 1987; Miller and von Flotow, 1989). Vibration control of a translating string has been studied by Yang and Mote (1991), and Chung and Tan (1995), based on the transfer function of the string including the dynamics of the sensing and actuation devices. The vibration suppression of the translating string was recently implemented by Lee and Mote (1996) through applying control at a boundary.

Boundary control of motions or forces at the ends of the beam has several advantages over control schemes acting within the domain. In boundary control, control laws can be obtained either from a global dynamic model of the system where a Lyapunov functional is the total vibration energy, or from a local dynamic model where boundary motion determines the excitation of a boundary controller. The control laws is not only easily implemented by active and passive means at the controlled boundary, but also the model of the beam is unaltered by the control sensors and actuators.

The subject of this study is to control the transverse motion of a translating tensioned Euler-Bernoulli beam via active or passive damping applied at a boundary. Generally, the gyroscopic and dispersive nature of the moving beam makes the design of control difficult. One of interesting phenomena observed in the translating continuum is that the total energy of free oscillation in the undamped translating beam is not constant. The variation in the vibration energy is due to energy flux between the translating continuum and the boundary support. In this paper, boundary control laws to stabilize the beam vibration without discretization are designed. Based on the wave expression proposed by Lee and Mote (1997), energy dissipated by control is quantified. The optimal damping coefficient maximizing the energy dissipation is also determined. Free vibration of the damped translating beam is simulated through the finite difference scheme for the comparison to the theoretical predictions.

2 System Model

The linear equation of transverse motion of a Euler-Bernoulli beam traveling at constant velocity \( v \) between two supports that are separated by a distance \( L \) is (Chubachi, 1958; Mote, 1965)

\[
\rho \left( \frac{\partial^2 W}{\partial t^2} + 2v \frac{\partial W}{\partial t} + v^2 \frac{\partial^2 W}{\partial x^2} \right) - S \frac{\partial^2 W}{\partial x^2} + EI \frac{\partial^4 W}{\partial x^4} = 0, \quad X \in (0, L)
\]

(1)

where \( W(X, T) \) is transverse displacement, \( \rho \) is linear density, \( S \) is tension, and \( EI \) is flexural rigidity of the beam. Introduction of the dimensionless variables

\[
x = \frac{X}{L}, \quad w = \frac{W}{L}, \quad t = \frac{T}{L^2 \sqrt{\frac{EI}{\rho}}},
\]

\[
v = \frac{VL}{\sqrt{\frac{EI}{\rho}}}, \quad P = \frac{\sqrt{EI}}{L^2}
\]

The transverse motion of a translating tensioned Euler-Bernoulli beam is controlled by passive or active damping applied at a boundary. Even for an undamped beam with a symmetric boundary configuration, the interaction between the translating continuum and the stationary or moving boundary leads to energy variation in free motion. With the time-varying energy chosen as a Lyapunov functional, boundary control laws are designed based on Lyapunov's second method. For various types of translating beams, energy dissipation by boundary damping is quantified using the method of traveling waves. The optimal value of damping, maximizing the energy dissipation, is also explicitly represented by system parameters. The analytical results are compared with numerical simulations using the finite difference scheme.
into (1) gives the normalized form
\[ w_x + 2w_{xx} + v^2 w_x - Pw_{xx} + w_{xx} = 0, \quad x \in (0, 1). \quad (2) \]

Here the subscript notation indicates partial differentiation. As shown in Fig. 1, the translating beam is controlled at the downstream boundary. The uncontrolled (upstream) boundary is either simply supported or fixed. The translating beam subjected to a stationary controller can be a model for axially moving systems of constant length such as tapes, band saws and belts which are usually coupled with an intermediate constraint [Fig. 1(a)]. As shown in Fig. 1(b), a translating controller is also applied to the moving boundary of the translating beam with time varying length, such as flexible manipulators with a prismatic joint and deploying or retracting of flexible appendages. The transverse velocity of a material particle on the translating beam in stationary coordinates is \( w_x + uw_x \). If linear viscous damping is located at the stationary boundary, the corresponding boundary condition is \( Pw_{xx} - w_{xx} = -dw \), where \( d \) is the damping coefficient. However if damping is applied on the moving boundary, it generates a restoring force proportional to the material velocity \( (w_x + uw_x) \).

Since the centrifugal force \( v^2 w_x \) in (2) is analogous to a compressive force to the beam, the critical transport speed for divergence instability is obtained from the time-independent terms in the equation. The critical speed is \( v_c = \sqrt{P + \pi^2} \) for a beam with simple supports and \( v_c = \sqrt{P + 4\pi^2} \) for fixed supports (Wickert and Mote, 1990). For sufficiently high tension, these critical speeds reduce to \( v_c \approx \sqrt{P} \), which is the critical speed of a translating string. In this paper, \( v \) remains constant and subcritical (\( |v| < v_c \)). The total mechanical energy \( E(t) \) of the beam is the sum of the kinetic and potential energies:
\[ E(t) = \frac{1}{2} \int_0^l (w_x + uw_x)^2 dx + \frac{1}{2} \int_0^l (Pw_x^2 + w_{xx}^2) dx. \quad (3) \]

3 Design of Control Laws

The objective of the boundary control is to dissipate vibration energy through either passive or active means applied at a boundary. The time-rate of change of energy is obtained through application of the one-dimensional transport theorem:
\[ \dot{E}(t) = E_t + v \dot{E}_t \quad (4) \]
where \( \dot{} \) is \( d/dt \), \( \dot{E}_t \) is \( \partial \dot{E}/\partial t \), and
\[ \dot{E} = \frac{1}{2} (w_x + uw_x)^2 + \frac{1}{2} (Pw_x^2 + w_{xx}^2) \quad (5) \]
is total energy density of the beam. The first term on the right side of (4) describes the local rate of change within the domain and the second term represents energy flux across the boundaries at any instant of time. Substitution of (3) into (4), and the use of (2) yields
\[ \dot{E}(t) = (Pw_x - w_{xx})(w_x + uw_x) + w_{xx}(w_x + uw_x) \quad (6) \]
The energy flux, due to the shear force \( Pw_x - w_{xx} \) and the bending moment \( w_{xx} \), are observed. At the unconstrained boundaries, the instantaneous transverse velocity is \( w_x + uw_x \) and the material derivative of the slope is \( w_{xx} + w_{xx} \). For fixed boundary supports, the energy rate becomes
\[ \dot{E}(t) = -uw_{xx}^2(0, t) + uw_{xx}^2(1, t). \quad (7) \]
This time-varying energy, even without any energy source or damping, results from the fact that the bending moment \( w_{xx} \) does work on the end of the beam with the non-zero convective term \( uw_x \) (Wickert and Mote, 1989). It is noted from (7) that energy is transferred out of the beam at \( x = 0 \) and into the beam at \( x = 1 \). For the beam subject to simple supports, the material particles enter the span with the instantaneous velocity \( uw_x \) under the shear force \( Pw_x - w_{xx} \). Over a cycle of oscillation, the sign and magnitude of energy transfered at the simply supports are equal to those at the fixed supports (Lee and Mote, 1997).

3.1 Control Laws at Downstream Boundary

Consider the case where the vertical motion of the translating beam at the downstream end is controlled. When the upstream boundary is fixed \( (w(0, t) = w_x(0, t) = 0) \), the energy rate from (6) becomes
\[ \dot{E}(t) = -uw_{xx}^2(0, t) + \{ Pw_x(1, t) - w_{xx}(1, t) \} [w_x(1, t) + uw_x(1, t)] \quad (8) \]
If the upstream boundary is simply supported \( (w(0, t) = w_x(0, t) = 0) \), the time-rate of energy is
\[ \dot{E}(t) = -\{ Pw_x(0, t) - w_{xx}(0, t) \} uw_x(0, t) + \{ Pw_x(1, t) - w_{xx}(1, t) \} [w_x(1, t) + uw_x(1, t)] \quad (9) \]
The first term of the right side of both Eqs. (8) and (9) is negative. The second term describes the rate energy produced by the undetermined boundary control. When control laws are designed to make the second term negative, the total energy is dissipated by the Lyapunov direct method. The following control laws are proposed ensuring \( \dot{E}(t) \leq 0 \):
- **linear velocity control,**
  \[ w_x(1, t) = -k w_x(1, t) \text{ Sgn } \{ Pw_x(1, t) - w_{xx}(1, t) \}, \quad k > v \quad (10) \]
- **linear slope control,**
  \[ w_x(1, t) = -k w_x(1, t) \text{ Sgn } \{ Pw_x(1, t) - w_{xx}(1, t) \}, \quad k > \frac{1}{v} \quad (11) \]
- **linear force control,**
  \[ Pw_x(1, t) - w_{xx}(1, t) = -k \{ w_x(1, t) + uw_x(1, t) \}, \quad k > 0 \quad (12) \]
Force control (12) represents linear viscous damping with restoring force proportional to material velocity \( w_0 + uw_x \). Therefore viscous damping with material velocity feedback or active control with direct velocity feedback can stabilize free vibration without an additional control at the boundary. For a deploying beam with time varying length, linear viscous damping attached to the moving boundary generates force control (12), because local velocity relative to a fixed observer becomes \( w_0 + uw_x \). In this paper, the last control law (12) is chosen.

3.2 Control Laws at Upstream Boundary. When the upstream boundary is controlled and the uncontrolled boundary \((x = 1)\) is fixed, the associated energy rate is similarly obtained from (6).

\[
\dot{E}(t) = \dot{w}_m^2(1, t) - \{Pw_x(0, t)
- w_{xx}(0, t) \} \{w(0, t) + uw_x(0, t)\}. \tag{13}
\]

Three control laws ensure negative energy flux at the controlled boundary and suppress the vibration energy. Linear force control, corresponding to control (12) at the downstream boundary, is

\[
Pw_x(0, t) - w_{xx}(0, t) = k_f \{w(0, t) + uw_x(0, t)\},
\]

\[
k_f > 0. \tag{14}
\]

The control law (14) represents the vertical restoring force by viscous damping attached on the boundary. Note that this boundary control does not always guarantee \( \dot{E}(t) = 0 \), because of the positive energy transfer at the downstream fixed boundary.

4 Wave Characteristics of Translating Beams

The sign of energy transferred by the interaction between the translating continuum and the boundary support is determined by calculating the energy rate \( \dot{E}(t) \). In this section, the magnitude of the energy transfer is calculated based on the traveling waves. The natural modes of vibration, generally represented by standing waves, are representable by the superposition of equal but opposite traveling waves. Therefore, energy in the natural vibration modes is divided into two energies contained in downstream and upstream traveling waves. Energy transferred at a boundary is then calculated by examining two traveling waves incident on and reflected from the boundary.

4.1 Traveling Wave Characteristics. At subcritical speeds, harmonic waves propagating along the translating tensioned beam (2) have the form

\[
w(x, t) = A_ie^{i(\omega t - kx)}.
\]

The dimensionless frequency \( \omega \) is related to the dimensional frequency \( \Omega \) by

\[
\omega = \Omega \sqrt{\frac{P}{Ei}}.
\]

The wavenumber, \( k \), satisfies the dispersion relation

\[
k^4 + (P - \nu^2)k^2 + 2\omega\nu k - \omega^2 = 0. \tag{17}
\]

Two of the four wavenumbers are real and two form a conjugate pair with a positive real part. Denote the four wavenumbers as \( k_d, -k_u, \) and \( k_u \pm ik_d \). Here \( k_d \) and \( k_u \) describe the wavenumbers of downstream and upstream propagating waves, and the complex wavenumbers describe evanescent (exponentially decaying) waves. Due to translation of the beam, the attenuating waves have a positive real part \( k_d \) and propagate downstream before fully decaying. \( k_u \) is asymptotic to zero as tension \( P \) increases and/or transport speed \( v \) decreases. The transverse displacement of the beam is then described by the four traveling waves:

\[
w(x, t) = A_d e^{-i(k_d x - \omega t)} + A_u e^{+i(k_u x - \omega t)}
+ A_d e^{-i(k_u x + \omega t)} + A_u e^{+i(k_u x + \omega t)}. \tag{18}
\]

Here \( A_d \) and \( A_u \) are the complex amplitudes of the downstream and upstream propagating waves, and \( A_d^* \) and \( A_u^* \) are the complex amplitudes of the evanescent waves. The phase velocities of propagating waves are \( c = \omega/k_d \) downstream and \( c = \omega/k_u \) upstream.

4.2 Energy Reflection Coefficient. When a traveling wave is incident on a boundary, both traveling and evanescent waves are produced. The evanescent waves with an imaginary or complex wavenumber decay exponentially and do not propagate wave energy. Exceptionally the waves are associated with energy flow only through the interaction between two opposite evanescent waves (Bobrovnitskii, 1992; Mead, 1994). In this study, the expression for energy transmitted by interacting with the boundary is determined by considering traveling waves. The energy contained in one wavelength \( \lambda = 2\pi/k \) of a harmonic traveling wave \( A_i e^{i(\omega t - kx)} \) is

\[
E_i = \int_0^{\lambda} \dot{E}dx = \pi(P + k^2)AA^* = \pi\omega \mathcal{Z} AA^*. \tag{19}
\]

where \( \mathcal{Z} = (P + k^2)/c \) is the mechanical impedance of the beam and \( c \) is the phase velocity of the wave. The asterisk denotes the complex conjugate. When the wave is incident on a boundary, the amplitude and phase of the wave reflected from the boundary is given by the reflection coefficient

\[
r = \frac{A_t}{A_i} = |r| e^{i\theta}, \tag{20}
\]

where \( A_i \) and \( A_t \) are the amplitudes of the incident and reflected traveling waves. The reflection coefficient \( r \) is easily derived from the boundary conditions. The energy \( \Delta W \), transferred at the boundary during wave reflection, equals the difference between the energies of the reflected and incident waves:

\[
\Delta W = E_r - E_i = \pi\omega \mathcal{Z} A_t A_t^* - \mathcal{Z} A_i A_i^*). \tag{21}
\]

Here \( \mathcal{Z} \) and \( \mathcal{Z}_t \) are the mechanical impedances of the reflected and incident waves. The energy reflection coefficient

\[
R = \frac{\Delta W}{E_i} = \frac{\mathcal{Z}_r}{\mathcal{Z}_t} rr^* \tag{22}
\]

is introduced to quantify the ratio of the reflected wave energy to the incident one (Lee and Mote, 1997). The different impedances, \( \mathcal{Z}_r, \mathcal{Z}_t \), lead to energy flux even in the undamped translating beam where the incident and reflected waves have the same amplitude (\(|r| = 1\)). The ratio of energy dissipated at the boundary to the incident wave energy is then \((1 - R)\).

5 Wave Reflection and Energy Dissipation

When a downstream traveling wave with amplitude \( A_t \) is incident on a boundary with force control (12), it is reflected into upstream propagating and evanescent waves. The wave response is then represented by

\[
w(x, t) = A_d e^{-i(k_d x - \omega t)} + A_u e^{+i(k_u x - \omega t)} + A_d e^{-i(k_u x + \omega t)} + A_u e^{+i(k_u x + \omega t)}, \tag{23}
\]

where \( A_d \) and \( A_u \) are the amplitudes of the reflected propagating and evanescent waves. Here the wavenumbers are determined by the dispersion relation (17). Substitution of (23) into the boundary conditions, \( w_x(1, t) = 0 \) and (12), gives the reflection coefficient

\[
r = \frac{A_t}{A_i} = \frac{N_i + iD_i}{D_i + iN_i}, \tag{24}
\]
where

\[ N_i = 2k_d k_i \}^2 + (P + k_\mu)k_\alpha - k_\beta \}
\]

\[ N_i = (k_i^2 - k_\beta^2) \}^2 + (P + k_\mu)k_\alpha - k_\beta \}
\]

\[ D_i = 2k_d (P + k_\mu)k_\alpha - k_\beta \}
\]

The energy reflection coefficient at the downstream boundary is then

\[ R_d = \frac{k_d (P + k_\mu)k_\alpha - k_\beta}{k_d (P + k_\mu)k_\alpha - k_\beta} \]

Figure 2(a) illustrates \( R_d \) for the cases of \( \omega = 1.500 \) and 1000 at tension \( P = 100 \) and \( v = 1 \). At \( \omega = 1 \), the boundary damping shows complete dissipation (zero reflection) of the incoming wave energy. At low frequency, the force \( w_{s,0} \) by bending stiffness becomes negligible comparing to \( P \). In this case, the damping coefficient \( (k_f = 10) \) for the maximal vibration suppression is actually identical to the optimal damping coefficient for a translating string. When the damping coefficient vanishes \( (k_f = 0) \) where the boundary support is essentially free, it is shown that \( R_d < 1 \). However, for \( k_f = \infty \) where the boundary is fixed, the sign of energy flux into the beam is reversed and energy is transmitted \( (R_d > 1) \), as predicted in (7). Figure 2(b) shows that all the energy of low frequency wave is suppressed by optimal damping. The energy dissipation increases as tension \( P \) increases (Fig. 2(c)). At high frequency or low tension where the effect of bending stiffness is dominant, an additional moment control is needed for complete energy dissipation of the incident wave.

When a traveling wave of amplitude \( A_i \) propagates on an upstream boundary with force control (14), the wave response is represented by

\[ w(x, t) = A_i e^{ik x + k_\mu t} + A_i e^{-ik x - k_\mu t} + A_i e^{-(ik x + k_\mu t)} + A_i e^{-(ik x - k_\mu t)} \]

Substitution of (26) into boundary conditions, \( w_{s,0}(0, t) = 0 \) and (14), gives the reflection coefficient \( r \) in (24). Here

\[ N_i = 2k_d k_i \}^2 + (P + k_\mu)k_\alpha - k_\beta \}
\]

\[ N_i = (k_i^2 - k_\beta^2) \}^2 + (P + k_\mu)k_\alpha - k_\beta \}
\]

\[ D_i = 2k_d (P + k_\mu)k_\alpha - k_\beta \}
\]

The energy reflection coefficient at the controlled boundary is then

\[ R_d = \frac{k_d (P + k_\mu)k_\alpha - k_\beta}{k_d (P + k_\mu)k_\alpha - k_\beta} \]

The energy reflection coefficient is plotted in Fig. 3 for incident waves of frequency \( \omega = 1.500 \) and 1000. It is noted that energy is transmitted into the damped beam \( (R_d > 1) \) at the controlled boundary with small damping coefficient \( k_f \). In this case, energy transferred by the translating medium at the boundary exceeds energy dissipated by the damping component. The energy reflection coefficients (25) and (27) are represented in terms of system parameters and wavenumbers. For the following special cases \( (EI = 0, v = 0 \) and \( P = 0) \), the energy coefficients at the controlled boundary are explicitly derived.

5.1 Case of \( EI = 0 \): Translating String. If the bending stiffness is negligible, the translating tensioned beam becomes a translating taut string. For a downstream wave propagating on boundary control at \( x = 1 \), the nondispersive motion of the string is represented by two traveling waves:

\[ w(x, t) = A e^{(i\omega - k_\mu) x} + A e^{(i\omega + k_\mu) x} \]

where the corresponding wavenumbers are

\[ k_d = \frac{\omega}{c_0 + v} \]

\[ k_d = \frac{\omega}{c_0 - v} \]

Here \( c_0 = \sqrt{\frac{E}{\rho}} \) is the phase velocity when \( v = 0 \). Substitution of (28) into the boundary condition \( P w_s(1, t) = -k_f \}^2 w_s(1, t) \) and use of (29) gives the reflection and energy reflection coefficients

\[ r = \left( \frac{c_0 - v}{c_0 + v} \right) \left( k_d c_0 - P \right) \]

\[ R_d = \left( \frac{c_0 - v}{c_0 + v} \right) \left( k_d c_0 + P \right) \]

5.2 Case of \( v = 0 \): Stationary Tensioned Beam. When \( v = 0 \), the wave motion of (23) for a downstream wave becomes

\[ w(x, t) = A e^{ik x + k_\mu t} + A e^{ik x - k_\mu t} + A e^{-ik x} e^{ik x} \]

\[ \text{JANUARY 1999, Vol. 121 / 21} \]

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Fig. 3 Energy reflection coefficient $R_0$ of a translating tensioned beam with boundary damping at $x = 0$ for the cases $\omega = 1,500$, and $10^4$ when $P = 1$ and $v = 1$.

where the wavenumbers from (17) with $v = 0$ are

$$k = k_\lambda = \left( \frac{-P}{2} + \frac{P^2}{4} + \omega^2 \right)^{1/2},$$

$$k_\lambda = \left( \frac{P}{2} + \frac{P^2}{4} + \omega^2 \right)^{1/2}, \quad k_\sigma = 0. \quad (32)$$

Then the reflection and energy reflection coefficients at $x = 1$,

$$r = -\frac{\omega k^3 - i \{ k_\lambda (k_\lambda^2 + k_\sigma^2) - k_\sigma \}}{\omega k^3 - i \{ k_\lambda (k_\lambda^2 + k_\sigma^2) + k_\sigma \}}, \quad R_0 = rr^*, \quad (33)$$

are obtained by substituting (31) into the boundary conditions of the controller, which are $P_{\nu_\lambda}(1, t) - w_{\nu_\lambda}(1, t) = -k_\lambda w_{\nu_\lambda}(1, t)$ and $w_{\nu_\lambda}(1, t) = 0$. A stationary beam with zero tension is the special case of the tensioned beam. When tension vanishes, $k = k_\lambda = \sqrt{\omega}$ and the reflection coefficient is simply represented in terms of input frequency and control gain:

$$r = -\frac{\sqrt{\omega} - i (2k_\lambda - \sqrt{\omega})}{\sqrt{\omega} - i (2k_\lambda + \sqrt{\omega})}. \quad (34)$$

For an upstream wave incident on boundary control at $x = 0$, the reflection and energy reflection coefficients of the stationary tensioned beam are identical to (33).

5.3 Case of $P = 0$: Translating Beam. For a translating beam with zero tension ($P = 0$), the dispersion relation

$$k^4 - (\omega - vk)^2 = 0 \quad (35)$$

gives the explicit expressions for the wavenumbers such as

$$k_\lambda = -\frac{v}{2} + \sqrt{\omega + \frac{v^2}{4}}, \quad k_\sigma = -\frac{v}{2} - \sqrt{\omega + \frac{v^2}{4}}, \quad (36)$$

The reflection coefficient by boundary control at $x = 1$ is given by the substitution of the wavenumbers in (36) into (24) and use of $P = 0$. The energy reflection coefficient is then

$$R_0 = \left( \frac{\omega}{2} + \sqrt{\omega + \frac{v^2}{4}} + \frac{v}{2} + \sqrt{\omega + \frac{v^2}{4}} \right)^3 \quad (37)$$

When the upstream boundary is controlled, the energy coefficient $R_0$ is similarly obtained from (27).

6 Optimal Damping for Maximal Energy Dissipation

When force control (12) is applied at $x = 1$, the best selection of the control parameter $k_\lambda$ is achieved by the minimization of the energy reflection coefficient (25). As shown in Fig. 2(a), a value of $k_\lambda$, maximizing the energy dissipation, exists and the optimal value is calculated from

$$\frac{dR_0}{dk_\lambda} = 0. \quad (38)$$

For the translating tensioned beam (2), the energy reflection coefficient (25) does not have explicit form with respect to $k_\lambda$. In this case, the optimal value of boundary control is numerically obtained. However, for the cases of $EI = 0$, $v = 0$ and $P = 0$, the optimal coefficient is explicitly determined.

6.1 Case of $EI = 0$. For the case of a translating string, the optimal damping coefficient,

$$k_{opt} = \frac{P}{c_0} = \sqrt{\omega}, \quad (39)$$

is calculated from the energy coefficients (30). The optimal value of $k_\lambda$ is identical to the impedance of $\mathcal{Z}_f$ of the string, and independent of frequency and transport speed.

6.2 Case of $v = 0$. From the energy coefficient (33) for the stationary tensioned beam, the optimal damping for a harmonic wave of frequency $\omega$ is

$$k_{opt} = \sqrt{\frac{P^2 + \omega^2}{(P^2 + 4\omega^2)^{1/2}}} \quad (40)$$

At low frequency or high tension, $k_{opt} \approx \sqrt{\omega}$ which is the optimal value for the translating string. At high frequency, where the effect of bending stiffness is dominant, the optimal gain (40) is asymptotic to the optimum $\sqrt{\omega/2}$, which is the optimal value for a stationary beam with zero tension. The optimal damping coefficients of stationary beams and string are shown in Fig. 4.
Table 1 Optimal damping for the special cases of the translating beam

<table>
<thead>
<tr>
<th>$k_{opt}$ at $x = 1$</th>
<th>$k_{opt}$ at $x = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EI = 0$</td>
<td>$\sqrt{P}$</td>
</tr>
<tr>
<td>$v = 0$</td>
<td>$\frac{\sqrt{P^2 + 2\omega^2}}{(P^2 + 4\omega^2)^{3/2}}$</td>
</tr>
<tr>
<td>$P = 0$</td>
<td>$\frac{v}{2} + \sqrt{\frac{v}{2}}$</td>
</tr>
</tbody>
</table>

6.3 Case of $P = 0$. The optimal damping coefficient for a translating beam with zero tension

$$k_{opt} = -\frac{v}{2} + \sqrt{\frac{\omega}{2}}.$$  \hspace{1cm} (41)

is calculated from $R_a$ in (37). When the upstream boundary is controlled, the optimal value is also obtained by

$$k_{opt} = \frac{v}{2} + \sqrt{\frac{\omega}{2}}.$$  \hspace{1cm} (42)

The discrepancy between the two optimal values is the magnitude of transport speed $v$. The optimal value (41) vanishes when input frequency satisfies $\omega = v^2/2$, and becomes negative for $\omega < v^2/2$.

6.4 Translating Tensioned Beam. Table 1 summarizes the optimal damping coefficients for the above three cases. For the translating tensioned beam, the optimal damping coefficient is numerically calculated. The optimal damping at the downstream boundary is shown in Fig. 5 together with the optimal values for the cases of $EI = 0$ and $P = 0$. At high frequency, the optimal coefficient is close to (41) for the case of $P = 0$.

At low frequency, the optimal value is essentially identical to that for a translating string ($k_{opt} = \sqrt{P}$). The optimal value of damping at the downstream boundary decreases with increasing speed $v$. The optimal value of control (14) located at the upstream boundary is shown in Fig. 6 for $v = 1, 5$ and $v = 10$. In these cases, the optimal value increases with $v$.

This analysis of optimal control for the translating beam is useful for the design of stabilizing boundary control using either a passive damping or active control. Especially, if a dominant frequency of excitation along the beam is known or a specific frequency component is to be controlled, the frequency-dependent expression of the optimal damping is easily used for the selection of control parameter $k_r$.

Consider the translating tensioned beam with the optimal damping at a boundary and a simple support at the other boundary. As shown in Fig. 2, boundary control with optimal damping can dissipate all incident waves at high tension. In this case, the time $t_c$ required for a wave of frequency $\omega$ to be dissipated is

$$t_c = \frac{1}{c_d} + \frac{1}{c_u} = \frac{1}{\omega} (k_d + k_u).$$  \hspace{1cm} (43)

where the wavenumbers $k_d$ and $k_u$ are calculated from the dispersion relation of the beam. Therefore, free motion of the high tensioned translating beam will be approximately stabilized within the time $t_c = (1/\omega_1)(k_d + k_u)$. Here $\omega_1$, $k_d$, and $k_u$ are the natural frequency, downstream and upstream wavenumbers of the fundamental vibration mode.

It is noted that the assumption of perfect control by optimal damping is valid for high tension or low frequency waves where
the effect by bending stiffness becomes negligible. At low tension or high frequency, an additional moment control at the controlled boundary is required for complete dissipation of vibration energy. Unlike force control, moment control dissipates high frequency waves very well.

7 Numerical Simulation

The analytically predicted optimal control of a translating tensioned beam is compared to results of numerical experiments. An implicit difference operator is used because the fourth order term \( w_{xx} \) requires a small time increment in an explicit method. A unit length of the beam is divided into \( n \) equal intervals \( h = 1/n \), and the time increment is \( \kappa \). The displacement and velocity at each point are then

\[
 u_j = w(ih, j\kappa), \quad v_j = w_t(ih, j\kappa).
\]

For \( w_r = -2\omega w_{xx} + (P - v^2)w - w_{xxx} \), the difference equations for \( u_{j+1}^{(1)} \) and \( v_{j+1}^{(1)} \) are

\[
\begin{align*}
 v_{j+1}^{(1)} - v_{j}^{(1)} & = -u_{j+1}^{(1)} - u_{j}^{(1)} + \frac{(P - v^2)}{h^2} u_{j+1}^{(1)} - \frac{2u_{j+1}^{(1)} - u_{j}^{(1)}}{h^2} \\
 & \quad - \frac{u_{j-1}^{(1)} - 4u_{j}^{(1)} + 6u_{j+1}^{(1)} - 4u_{j+2}^{(1)} + u_{j+3}^{(1)}}{h^4},
\end{align*}
\]

\[
 u_{j+1}^{(1)} = u_{j}^{(1)} + \kappa v_{j+1}^{(1)}, \quad u_{j+1}^{(1)} = u_{j}^{(1)} + \frac{\kappa}{2}(v_{j+1}^{(1)} + v_{j}^{(1)}).
\]

The velocity \( v_{j+1}^{(1)} \) uses an implicit predictor \( u_{j+1}^{(1)} \) of the displacement for an improved estimation. Then \( u_{j+1}^{(1)} \) is estimated from \( v_{j+1}^{(1)} \) and \( v_{j+2}^{(1)} \). A mesh number \( n = 400 \) and a time step \( \kappa = 5 \times 10^{-6} \) are used for stable numerical solutions. These spatial and time steps were chosen following comparisons of the numerical predictions of the beam response without boundary control to the known exact solution.

For a translating tensioned beam with frequency \( \omega = 3\pi \), tension \( P = 100 \), and transport speed \( v = 1 \), the analytically predicted optimal coefficient and the maximum stabilizing time are

\[
k_{opt} = 9.94, \quad t_{m} = \frac{1}{\omega} (k_{d} + k_{c}) = 0.2011
\]

where \( k_{d} = 0.8540 \) and \( k_{c} = 1.0409 \). Numerical solution of the total energy of the beam with damping at \( x = 1 \) and a simple support at \( x = 0 \) is shown in Fig. 7. The initial conditions are \( w(x, 0) = 0.1 \sin (3\pi x) \) and \( w_t(x, 0) = 0 \). As predicted in Section 6, the initial vibration energy is best dissipated by damping with the optimal coefficient \( k_{opt} = 9.94 \). The time for the energy to be controlled is in accord with the predicted value \( t_{m} \) of 0.2011. For the case where \( k_{d} = k_{opt} \), wave energy reflected from the boundary damping and the maximum time for stabilization increase with the difference between \( k_{d} \) and \( k_{opt} \). An uncontrolled translating beam with simple supports at both boundaries exhibits a small variation of its energy around a constant value (100 percent).

8 Conclusions

In this paper, vibration control of transverse motions in a tensioned translating beam has been implemented by optimal damping applied at a boundary.

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References


