Vibration Control of an Axially Moving String by Boundary Control

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The stabilization of the transverse vibration of an axially moving string is implemented using time-varying control of either the boundary transverse motion or the external boundary forces. The total mechanical energy of the translating string is a Lyapunov functional and boundary control laws are designed to dissipate the total vibration energy of the string at the left and/or right boundary. An optimal feedback gain determined by minimizing the energy reflected from the boundaries, is the ratio of tension to the propagation velocity of an incident wave to the boundary control. Also the maximum time required to stabilize all vibration energy of the system for any initial disturbance is the time required for a wave to propagate the span of the string before hitting boundary control. Asymptotic and exponential stability of the axially moving string under boundary control are verified analytically through the decay rate of the energy norm and the use of semigroup theory. Simulations are used to verify the theoretically predicted, optimal boundary control for the stabilization of the translating string.

1 Introduction

Vibration of axially moving continua limits their utility in many mechanical systems such as magnetic tapes, band saws, power transmission chains and belts, thread and fiber winders, paper-handling machinery, aerial cable tramways, and pipes containing fluids. Hence, vibration control of axially moving systems is needed to improve the performance and productivity of such mechanical systems, in particular, high speed precision systems.

Previous studies of vibration control have focused on stabilization of the translating system by passive damping and stiffness (Wickert and Mote, 1988). Adaptation to precise, time varying conditions at high translating speed is facilitated by active vibration control using external forces. But the distributed and gyroscopic properties of the axially moving continua make a stabilizing control design difficult in both theory and practice.

Over the last decade there has been a growing interest in the analysis and control of distributed parameter systems. Use of discrete modes when formulating active control of flexible systems, can lead to destabilization of the system via spillover (Balas, 1978).

Control methods based on a continuous model of the system can be separated into three classes. Distributed or point control forces applied in the domain of the system is the first class. However, distributed controllers are not practical to build, and the distributed system becomes uncontrollable and unobservable when point actuators and sensors are located at nodal points (Jai and Pritchard, 1987). In the second class, system parameter variation, like control of tension in a string or beam, can be used to control vibration (Habib and Radcliffe, 1991; Fujimoto et al., 1993; Rahn and Mote, 1994). The method is called parametric control or stiffness control. But the coupling of longitudinal and transverse motions inherent in tension variation complicates the design of a stabilizing controller for axially moving materials.

The third class reduces vibration energy in continuous systems using either boundary control of motions or external applied forces. The boundary control method provides a practical control design, because vibration is more easily controlled through a boundary point than using point actuators and sensors away from the boundaries. Control inputs to boundary control are determined based only on the local dynamics boundary motion. Moreover, boundary control can be implemented not only by active control such that a direct velocity feedback or power control, but also through a passive design using a damping mechanism. Boundary control of flexible systems has become an important research area recently. This idea was first applied to a string by Chen (1979), and then to an Euler-Bernoulli beam (Chen et al., 1987) and to a Timoshenko beam (Kim and Renardy, 1987). Recently the boundary control technique has been applied to the stabilization of nonlinear beams and plates (Lagnese, 1991).

Active vibration control via traveling wave theory dissipates vibration energy in flexible systems through wave absorption. The wave control method has been applied to control of beams (Vaughan, 1968) and has been developed for large space structures (von Flotow, 1986). Also vibration control by cancellation of traveling waves in various distributed parameter systems has been studied (Hagedorn, 1985; Mace, 1987). The wave control concept has been extended to energy flux flow control for uncertain flexible structures (MacMartin and Hall, 1991).

Two papers discuss active vibration control of the axially moving string using a distributed transfer function method originated by Butkovskiy (1983). Yang and Mote (1991) developed the transfer function of a closed-loop system consisting of the axially moving continuum, a point controller and the dynamics of point sensing and actuation devices. Further, they designed a stabilizing controller based on analysis of the root loci of the closed-loop system. The transfer function approach was also used to stabilize an axially moving string with a point sensor and a boundary actuator in the frequency domain (Chung and Tan, 1993).

In this paper, the boundary control technique is first applied to control the transverse vibration of an axially moving string. The time-optimal control gain for the maximum energy dissipation of the string under boundary control is determined by minimizing the energy of waves reflected from the boundaries. The
maximum time required to dissipate all vibration energy of the translating string for any initial conditions is also determined. Asymptotic and exponential stability of the string through boundary control is proven by use of an invariance principle and semigroup theory. Simulation results verify the effectiveness and optimality of the boundary control design to the stabilization of the axially moving string.

2 System Model

The linear equation governing the transverse motion \( W(X, T) \) of an axially moving string is

\[
\rho(W_{TT} + 2VW_{XT} + V^2W_{XX}) - PW_{XX} = F(X, T)
\]

where \( \rho \) is linear density of the string, \( V \) is axial transport speed of the string, \( P \) is tension and \( F(X, T) \) is the external distributed force. Introduction of the following dimensionless variables in (1)

\[
x = \frac{X}{L}, \quad w = \frac{W}{L}, \quad t = T\sqrt{\frac{P}{\rho L^2}},
\]

\[
v = V\sqrt{\frac{P}{P}}, \quad f(x,t) = F(X,T)\frac{L}{P}
\]

gives the normalized equation of motion and initial conditions:

\[
w_t(x, t) + 2vw_x(x, t) - (1 - v^2)w_{xx}(x, t) = f(x, t), \quad x \in (0, 1)
\]

\[
w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x).
\]

In this paper, the axially moving string is controlled at the left and/or right boundary. A schematic diagram of an axially moving string with an active boundary control (a) or a passive boundary damper (b) at the right boundary is shown in Fig. 1. In the right boundary controlled string, the boundary at the left end is pinned and the right boundary motion is specified to dissipate the energy of vibration. The total mechanical energy \( E(t) \) of the string becomes

\[
E(t) = \frac{1}{2} \int_0^1 w_t^2 dx + \frac{1}{2} \int_0^1 (w_t + vw_x)^2 dx.
\]

A critical speed at \( v_c = 1 \) exists at which the fundamental natural frequency vanishes and a divergence instability occurs (Mote, 1965). In this study, \( v \) is a subcritical speed, \( |v| < 1 \).

3 Boundary Control Law

The objective of the control system is to stabilize asymptotically the transverse vibration of the axially moving string with a time-varying boundary condition at \( x = 0 \) and/or at \( x = 1 \). The positive definite, total energy \( E(t) \) in (3) is a Lyapunov functional candidate. Because the spatial variable \( x \) is a function of time, the time rate of change of \( E(t) \) is a material derivative:

\[
\dot{E}(t) = \dot{E}_c + vE_c|_0^1.
\]

where \( (\dot{\cdot}) = d/dt, \quad (\cdot)_t = \partial/\partial t \). The first term on the right hand side describes the local rate of change inside the region and the second term represents energy flux at the boundaries. Substitution of (3) into (4), and use of (2), yields

\[
\dot{E}(t) = vww_t^2(x, t)|_0^1 + w_t(x, t)w_x(x, t)|_0^1.
\]

Even though transverse velocity \( w_t \) at each boundary is zero, the instantaneous transverse velocity of a material particle at the boundary is \( vw_t \). Consequently, at each boundary, the transverse component of the string tension \( w_t \) does work on the string. The first term on the right in (5) represents energy flux at each boundary which is transverse velocity multiplied by string tension (Wickert and Mote, 1989). For the translating string with fixed boundary conditions, any traveling wave impinging on the boundaries causes the decay of the total energy at \( x = 0 \) and the increase of the energy at \( x = 1 \). Also the second term of (5) describes the energy flux produced by an as yet undetermined boundary controller at the boundaries. Then Lyapunov's direct method is used to choose control laws for the stabilization of the vibration energy.

3.1 Right Boundary Control Law. The time derivative of the energy in the string with a right boundary controller at \( x = 1 \) and the fixed condition at left boundary, becomes

\[
\dot{E}(t) = -vw_t^2(0, t) + w_t(1, t)\{vw_t(1, t) + w_t(1, t)\}.
\]

There are three candidates for the stabilization of the system. The following control laws make the rate of change of the total energy negative semidefinite, \( \dot{E}(t) \leq 0 \):

velocity control law:

\[
w_t(1, t) = -kw_t(1, t), \quad k_t > v
\]

force control law (local velocity feedback):

\[
w_t(1, t) = -kw_t(1, t), \quad 0 < k_f < \frac{1}{v}
\]

force control law (material velocity feedback):

\[
w_t(1, t) = -k_f\{vw_t(1, t) + w_t(1, t)\}, \quad k_f > 0
\]

The slope measurement \( w_t(1, t) \) is used as input to a velocity controller (7) to dissipate energy. The two control laws (8) and (9) stabilize the vibration by force control proportional to local velocity or material velocity at the boundary. Comparing (7) with (8), we easily obtain \( k_t = 1/k_f \). Also \( k_f = k_f/1 + uk_f \) from (8) and (9). The tension \( w_t \) in (8) and (9) is analogous to a transverse control force \( f(x,t) \) applied at the boundary. From Hamilton's principle for systems of changing mass (McIver, 1973),

\[
\int_0^1 (\delta L + \delta W_v + \delta W_e) dt = 0
\]

where

\[
L = T - U = \frac{1}{2} \int_0^1 (w_t + vw_x)^2 dx - \frac{1}{2} \int_0^1 w^2 dx
\]

\[
\delta W_v = f_t(t)\delta w(1, t)
\]

\[
\delta W_e = -v\{w_t(1, t) + uvw_t(1, t)\}\delta w(1, t).
\]
W_t and W_v are nonconservative work terms by the control force \( f_c(t) \) and the right boundary, respectively. From (10), the relation between the force \( f_c(t) \) and the tension on the right boundary is

\[
w(x, t) = w_c(1, t) = f_c(t).
\]

Finally, from (8) and (11), the force control law (8) with a local velocity feedback becomes

\[
f_c(t) = w_c(1, t) = -k_w w(1, t), \quad 0 < k_w < \frac{1}{v}.
\]

And the control law (9) is

\[
f_c(t) = w_c(1, t) = -k_w w(1, t), \quad k_d > 0.
\]

The control force \( f_c(t) \) can be implemented through a passive viscous damping design as well as active control methods like a direct velocity feedback control. Thus, one boundary controller with either a feedback gain \( 0 < k_w < 1/v \) for a local velocity feedback (12) or a gain \( k_d > 0 \) for a material velocity feedback (13) can stabilize the vibrating string without other apparatus.

### 3.2 Left Boundary Control Law

The rate of change of the energy for the string controlled at \( x = 0 \) and having a fixed boundary condition at \( x = 1 \) becomes

\[
E(t) = v w^2(1, t) - w(0, t) \left( w(0, t) + w(0, t) \right).
\]

The following control laws are proposed for the stabilization of the left boundary controlled string by making the second term on the right-hand side in (14) negative semidefinite:

- velocity control law:
  \[ w_c(0, t) = -k_w w(0, t), \quad k_w > v, \]
- force control law (local velocity feedback):
  \[ f_c(t) = -w_c(0, t) = -k_w w(0, t), \quad k_f < -\frac{1}{v} \quad \text{or} \quad k_f > 0 \]
- force control law (material velocity feedback):
  \[ f_c(t) = -w_c(0, t) = -k_d w(0, t) + w(1, t), \quad k_d > 0 \]

where \( f_c(t) \) is a transverse force applied at \( x = 0 \). The relation \( f_c(t) = -w_c(0, t) \) between the string tension and the control force is obtained by the same principle (10) used in determining right force control laws. We note from (16) that a negative damping force with a control gain \( k_f < -1/v \) as well as a positive damping force with a gain \( k_f > 0 \) will control the translating string.

The first term in (14) is always positive. Thus any wave traveling toward \( x = 1 \) causes an increase in the string energy \( E(t) \) by energy flux. A local increase in energy \( E(t) \) can be shown under the left boundary control (Fig. 6), even though the left boundary control stabilizes the string globally. The right boundary control is recommended for the control of the string, because the energy \( E(t) \) under right boundary control is a nonincreasing function.

### 4 Optimal Energy Dissipation

The maximum rate of energy decay from the translating string under boundary control is obtained by minimizing the energy reflected at the boundaries. Harmonic transverse waves \( w(x, t) \) in an axially moving infinite string can be represented by

\[
w(x, t) = A e^{i \omega (x - ct)}
\]

where \( \omega = \gamma c \) is the radial frequency of wave, \( \gamma \) is the wavenumber, and \( c \) is the phase velocity. The propagation velocity \( c \) is \( 1 - v \) in the upstream direction and \( 1 + v \) in the downstream direction for a string translating at speed \( v \). Due to the difference between the upstream and downstream wave velocities, the wavenumber \( \gamma \), of downstream is different from the upstream wavenumber \( \gamma \), for the same radial frequency \( \omega \):

\[
\omega = \gamma c (1 + v) = \gamma c (1 - v).
\]

As the transport speed \( v \) approaches the critical speed \( c = 1 \), the wavenumber \( \gamma \), of upstream waves goes to infinity, which can cause a large amplitude reflection of a small incident wave propagating to the left boundary (Section 4.2). The harmonic solution of (2) becomes

\[
w(x, t) = A_r e^{i \gamma c (1 + v) t} + A_i e^{i \gamma c (1 - v) t}.
\]

This represents harmonic motions of two traveling waves, one propagating in the positive \( x \) direction with an amplitude \( A_r \) and the second propagating in the left direction with an amplitude \( A_i \). In order to determine the wave reflection at the right or left boundary, the system is treated as a semi-infinite string with \( x = 0 \) for a right boundary controller and \( x = 0 \) for a left boundary controller.

### 4.1 Wave Reflection at Right Boundary Control

Any wave propagating in the positive \( x \) direction can be considered an incident wave to the right boundary (Graff, 1975). Therefore, the constant \( A_i \) is the amplitude of the incident wave at the right boundary. \( A_i \) represents the amplitude of the wave reflected from the boundary controller. Substitution (20) into the force control law \( w_c = -k_w w(1, t) \) gives

\[
A_i / A_r = -k_f (1 + v) - 1 / (1 + v).
\]

Note that the amplitude ratio (21) includes the wavenumber ratio, unlike the case of a stationary string. By use of (19) and defining \( M = -(k_f (1 + v) - 1)/(k_f (1 - v) + 1) \), the ratio (21) becomes

\[
A_i / A_r = M \left[ \frac{1 - v}{1 + v} \right].
\]

The ratio, depending on control gain \( k_f \) and transport velocity \( v \), is independent of the frequency \( \omega \) due to non-dispersive nature of the string. The absolute value \( |M| \) is less than 1 for the region of stable control gain \( 0 < k_f < 1/v \). The value \( M \) for various control gains \( k_f \) is shown in Table 1. Because \( M \) is positive when \( 0 < k_f < 1/(1 + v) \), the incident and reflected waves at \( x = 1 \) are in phase. The negative \( M \) when \( 1/(1 + v) < k_f < 1/v \) results in a reflected wave of opposite phase to the incident wave. When \( |M| = 1 \), the phase shift is either 0 or \( \pi \), depending on \( k_f \). The similar analysis for the force control law (11) with a material velocity feedback is also shown in Table 1 using \( k_f = k_f (1 - v k_f) \).

The difference in wavenumbers \( \gamma_i, \gamma_r \) for an incident wave, as well as the term \( M(k_f) \), makes the amplitude \( A_i \) of a reflection wave smaller than the amplitude \( A_r \) of an incident wave (Fig. 2). As the speed \( v \) closes to the critical speed \( c = 1 \), the amplitude \( A_i \) of the reflection wave goes to zero, causing a divergence instability by losing the propagation characteristics.

The system controlled by force control (9) with a material velocity feedback gives the following amplitude ratio:

\[
A_i / A_r = \frac{k_f - 1}{k_f + 1} \left[ \frac{1 - v}{1 + v} \right].
\]

The value \( ((k_f - 1)/(k_f + 1)) \) is also less than 1 for the stable control region, \( k_f > 0 \).
4.2 Wave Reflection at Left Boundary Control. Similarly, the amplitude ratio of the reflected wave to an incoming wave for the left boundary controlled string is obtained by putting (20) into the left boundary control law:

\[ f_c(t) = -w_0(0, t) = -k_d w(0, t) \]

\[ A_r = \frac{-k_d(1 - v) - 1}{k_d(1 + v) + 1} \left( \frac{1}{1 - v} \right). \]  

(24)

As the transport speed \( v \) increases to the critical speed \( c_s = 1 \), the amplitude \( A_r \) of a reflected wave goes to infinity. If the transport speed is less than the critical speed, but sufficiently high, then an instability occurs even for boundary control with the gain in the stable region \( k_d > 0 \) or \( k_d < -1/v \) due to the large amplitude of a reflected wave. From this point of view, force control at the upstream boundary can cause a failure in the linear model of a high speed translating string.

For a nondispersive medium, the velocity of energy transmission equals the phase velocity of the medium. Also if more energy flows across the outlet than flows into the domain, then the energy contained in length of the medium diminishes (Achenbach, 1975). Thus the energy in the reflected wave from the left boundary is less than that in the incoming wave, in spite of the large reflection amplitude. This occurs because the propagation speed \((1 + v)\) of the reflection wave is greater than the speed \((1 - v)\) of the incident wave (Fig. 3).

The amplitude ratio for the string controlled by force control (17) with a material velocity feedback becomes

\[ A_r = \frac{k_d - 1}{k_d + 1} \left( \frac{1}{1 - v} \right). \]

(25)

4.3 Optimal Control Gain. From Eq. (21), the optimal gain \( k_f \) to maximize energy dissipation at \( x = 1 \) is

\[ k_f = \frac{1}{1 + v}. \]

(26)

In this case, no reflected wave is generated and all vibration energy in the incident wave is completely dissipated at the right boundary. The optimal control gain for the right boundary velocity control (7) is \( k_r = 1 + v \). For the original equation of motion (1), the optimal gain of force control \( f_c(t) = -k_f w(1, t) \) is \( k_f = P/\rho \), that is the ratio of tension to the propagation velocity of an incident wave, where \( \rho = \sqrt{P/\rho} \). When the string is stationary \( (v = 0) \), the optimal gain \( k_f = 1 \) was obtained by Glower (1995) using impedance matching. Similarly, the optimal control gain for the left boundary control from Eq. (23) is

\[ k_f = \frac{1}{1 - v}. \]

(27)

From (23) and (25), the optimal gain for the string controlled by force control with a material velocity feedback is

\[ k_f = 1. \]

(28)

The optimal gain is not only independent of the transport speed \( v \), but is also the same for both right and left boundary controls. Also the optimal gain for the original Eq. (1) becomes \( k_f = P/\rho \), that is the impedance of the string. Optimal gains of various boundary controls, resulting in the complete dissipation of incoming energy, are shown in Table 2 for the original equation of motion (1) as well as for the normalized system (2).

For force control provides a passive design in addition to active control. It is remarkable that vibration of an axially moving string can be perfectly stabilized by placing a viscous damping at \( x = 0 \) and/or at \( x = 1 \). Thus the boundary controller is called a perfect wave absorber.

Boundary control tuned to the optimal gain is also the time-optimal control of the vibrating string, because the time required to null vibration by boundary control can not be reduced by adding other boundary controllers. Consider the addition of controllers within \( 0 < x < 1 \) so that a propagating disturbance encounters control prior to reaching the boundary. Either or both

| Table 1 Amplitude ratio \( M \) by control gain \( k_f \) of right boundary control |
|-----------------|-----------------|-----------------|-----------------|
| \( k_f \) | \( k_d = 0 \) | \( k_d = 0 \) | \( k_d = 0 \) |
| \( M \) | \( 1 \) | \( 1 \) | \( 1 \) |
| \( E(t) \) | \( E(t) = 0 \) | \( E(t) < 0 \) | \( E(t) < 0 \) |

![Fig. 2 Reflected wave at x = 1 by right boundary control with different control gains k_f and transport speed v = 0.5](image)

![Fig. 3 Reflected wave at x = 0 by left boundary control with different control gains k_f and transport speed v = 0.5](image)
reflected and transmitted waves arise from the discontinuity of control. Because both the reflected and transmitted waves can not be stabilized simultaneously at the discontinuity, their simultaneous presence increases the maximum time required for the stabilization of the propagating waves. Thus the boundary control with an optimal control gain is time-optimal control for the stabilization of the vibrating string.

4.4 Maximum Time for Stabilization. The time required to reduce the vibration energy in the string to zero equals the maximum time for any initial disturbance wave to reach boundary control. This is the time required for a disturbance to propagate the length of the string upstream plus downstream. The maximum time $t_s$ required to absorb all the energy of the string by the right boundary control with an optimal gain for any initial condition is

$$ t_s = \left( \frac{1}{1 - v} + \frac{1}{1 + v} \right) T $$

where $T$ is the time required for waves to propagate along a stationary string with the length $L$, i.e., $T = L/c_0 = \sqrt{\rho L^2/\sigma}$. Note that the maximum time for stabilization of the string by the left boundary control with the optimal gain is the same as (29) by the right boundary control. If both boundaries are controlled with optimal control gains, the maximum time for the stabilization of the system is

$$ t_s = \left( \frac{1}{1 - v} \right) T $$

This is the time for an upstream wave starting at $x = 1$ to reach the left boundary control. The times for the stabilization of two typical axially moving materials, an aluminum string and a magnetic tape, by right or left boundary control are shown in Table 3.

Because boundary control with a nonoptimal gain generates reflected waves, the time needed to suppress the string vibration through the nonoptimal control is greater than the maximum times for the stabilization (29) or (30). Theoretical proofs for the asymptotic and exponential stability of the string under nonoptimal control are presented in the next section.

5 Stability

The boundary control laws (7)–(9) show that $\dot{E}(t)$ is negative semidefinite, ensuring stability but not asymptotic stability. In this section, we prove asymptotic stability, and furthermore exponential stability of the axially moving string under any right boundary force control (12) with the stabilizing control gain $0 < k_r < 1/v$. We need definitions of the energy norm and the semigroup for the distributed system as preliminaries.

Let the function space $H$ be

$$ H = \{ [w, w_1] \mid w \in H^1_0, w_1 \in L^2 \} $$

with the spaces $L^2$ and $H^1_0$ defined as

$$ L^2 := \{ f : [0, 1] \to H \int_0^1 f^2 dx < \infty \} $$

$$ H^1_0 := \{ f \in L^2 \mid f, f', \ldots, f^{(n)} \in L^2, f(0) = 0 \} $$

Then, the equation of motion (2) of a string controlled by the right boundary force (12) can be written in the distributed state form using the state vector $z(t) = [w, w_1]^T \in H$:

$$ z(t) = Az(t), \quad z(0) = z_0 \in H $$

where the operator $A : H \to H$ is defined as

$$ A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{(1 - v^2)\partial_x} - 2v\partial_x \end{bmatrix} $$

Also the domain $D(A)$ of the operator $A$ for the system controlled by the right boundary force control (12) is

$$ D(A) := \{ [w, w_1]^T \mid w \in H^1_0, w_1 \in H^1_0 \} $$

Then the following energy inner product is defined in $H$:

$$ \langle z_1, z_2 \rangle := \frac{1}{2} \int_0^1 w_1 w_2 dx + \frac{1}{2} \int_0^1 (w_1 + v w_1)(w_2 + v w_2) dx \quad (36) $$

where $z_1 = [w_1, w_1]^T$ and $z_2 = [w_2, w_2]^T$. We note that the space $H$, together with the energy inner product (36) becomes a Hilbert space. The energy norm induced by the energy inner product (36) is the total energy (3) of the string, i.e.,

$$ E(t) := \langle z, z \rangle = \langle z(t) \rangle $$

$$ = \frac{1}{2} \int_0^1 w_1^2 dx + \frac{1}{2} \int_0^1 (w_1 + v w_1)^2 dx $$

Now we represent the solution $z(t)$ of the system using a semi-

<table>
<thead>
<tr>
<th>Material:</th>
<th>Aluminum</th>
<th>Magnetic Tape</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length:</td>
<td>$L = 2$ m</td>
<td>$L = 0.3$ m</td>
</tr>
<tr>
<td>Area:</td>
<td>$\pi (6 \text{ mm})^2/4$</td>
<td>$A = 28 \mu m \times 12.7$ mm</td>
</tr>
<tr>
<td>Mass/length:</td>
<td>$\rho = 0.314$ kg/m</td>
<td>$\rho = 5 \times 10^{-7}$ kg/m</td>
</tr>
<tr>
<td>Tension:</td>
<td>$P = 20$ N</td>
<td>$P = 0.4$ N</td>
</tr>
<tr>
<td>Wave velocity:</td>
<td>$c_0 = \sqrt{\rho} = 7.98$ m/s</td>
<td>$c_0 = 28.3$ m/s</td>
</tr>
<tr>
<td>Dimensional time:</td>
<td>$T = \frac{L}{c_0} = 0.25$ s</td>
<td>$T = 0.01$ s</td>
</tr>
<tr>
<td>Time for stabilization ($v = 0.5$):</td>
<td>$t_s = 0.66$ s</td>
<td>$t_s = 0.027$ s</td>
</tr>
</tbody>
</table>
group \(S(t)\). For the system (34), \(A\) is dissipative on \(\mathcal{H}\) by \(\dot{E}(t) \equiv 0\). Then \(A\) generates a \(C_0\) semigroup \(S(t) = e^{tA}\) on \(\mathcal{H}\) (Chen, 1978; Mörgul, 1992), where \(S(t)\) is a bounded linear operator on \(\mathcal{H}\) for \(t \geq 0\). Finally, if \(z_0 \in D(A)\), then the system (34) admits a unique solution \(z(t) \in D(A)\) such that

\[
z(t) = S(t)z_0, \quad t \geq 0.
\]

(38)

### 5.1 Asymptotic Stability

The closed-loop system (34) is shown to be asymptotically stable using the invariance principle for function spaces (Henry, 1981):

Let \(V\) be a Lyapunov functional on \(\mathcal{H}\) such that \(\dot{V}(z) \equiv 0\) \(\forall z \in \mathcal{H}\). And define the set \(B = \{z \in \mathcal{H}|V(z) = 0\}\) with \(B_0\) the maximal invariant subset of \(B\). Then \(z(t) = S(t)z_0 \rightarrow B_0\) as \(t \rightarrow \infty\) for all \(z_0 \in \mathcal{H}\), where \(S(t)\) is the semigroup generated by the closed-loop system.

Using this invariance principle, we prove the following theorem governing asymptotic stability of the axially moving string by the right boundary force control.

**Theorem 1:** Consider the system (34). Then the solution \(z(t)\) of the system is asymptotically stabilized, \(\lim_{t \rightarrow \infty} z(t) = 0\), by the right boundary force control law (12).

**Proof:** With \(E(t)\) as a Lyapunov functional on \(\mathcal{H}\). The subset \(B\) of \(\mathcal{H}\) is

\[
B = \{ z(t) \in \mathcal{H}|\dot{E}(t) = 0 \}.
\]

From the invariance principle, asymptotic stability requires the maximal invariant subset of \(B\) to be the origin. The set \(B\) includes all \(z(t)\) with \(E(t) = 0\), including any \(z(t)\) satisfying \(w_s(0, t) = w_x(1, t) = 0\) from (6). Any solution \(z(t)\) of the system (34) asymptotically approaches to the maximal invariant set \(B_0\). Any element of \(B_0\), except zero solution, cannot be invariant, because waves of a nonzero solution propagate to \(x = 1\) at subcritical speed \(v < v_s = 1\) by the characteristics of wave equation. Therefore, the maximal invariant set includes only the origin, \(B_0 = \{0\}\), and \(z(t)\) is asymptotically stabilized by boundary control at \(x = 1\).

#### 5.2 Exponential Stability

To verify that the right boundary force control law (12) induces a rate of exponential decay of the energy \(E(t)\), the following theorem is first proved using an appropriate Lyapunov functional for the closed-loop system.

**Theorem 2:** Consider the system (2) under a boundary control law (12). Then there exists a \(T_2 > 0\), such that the energy \(E(t)\) given by (3) decays at \(\theta/(1/t)\) or faster for \(t \geq T_2\).

**Proof:** Define the functional:

\[
V(t) = 2(1 - \epsilon)tE(t) + \sigma(t)
\]

where \(\epsilon \in (0, 1)\) is an arbitrary constant and

\[
\sigma(t) = 2 \int_0^t x w_s(w_t + v w_x)dx.
\]

Equation (41) satisfies the following inequality

\[
|\sigma(t)| \leq \int_0^t (xw_t)^2dx + \int_0^t (w_t + v w_x)^2dx \\
\leq \int_0^t w_t^2dx + \int_0^t (w_t + v w_x)^2dx = 2E(t).
\]

Then, with use of (40) and (42), we obtain

\[
[2(1 - \epsilon)t - C]E(t) \leq V(t) \leq [2(1 - \epsilon)t+ C]E(t)
\]

for all \(t \geq 0\) and constant \(C \geq 2\). To make the functional \(V(t)\) positive definite for the worst case of (42), it follows that

\[
t > T_1 = \frac{C}{2(1 - \epsilon)}.
\]

(44)

Now we introduce

\[
\int_0^T \int_0^1 x w_x w_x dxdt = \int_0^T \int_0^1 x w_x^2 dxdt = \frac{1}{2} \int_0^T x w_x^2 (x, t) dt|_0^\infty.
\]

(46)

Substitution of the following identities

\[
\int_0^T \int_0^1 x w_x w_x dxdt = \frac{1}{2} \int_0^T x w_x^2 (x, t) dt|_0^\infty
\]

(47)

\[
\int_0^T \int_0^1 x w_x w_x dxdt
\]

(48)

into (45) yields the following form

\[
\sigma(T) = \sigma(0) - \int_0^T \int_0^1 x w_x^2 (x, t) dxdt + \int_0^T x w_x^2 (1, t) dt
\]

\[
-(1 - v^2) \int_0^T \int_0^1 x w_x^2 (x, t) dxdt
\]

\[
+ (1 - v^2) \int_0^T x w_x^2 (1, t) dt.
\]

(49)

(50)

Also differentiation of (50) with respect to time and use of the right boundary force control law (12), gives

\[
\sigma(t) = -\int_0^t x w_x^2 dx - (1 - v^2) \int_0^t x w_x^2 dx
\]

\[
+ (1 + k_1(1 - v^2)) w_x^2 (1, t).
\]

(51)

Then, there exists a constant \(T_3 > T_1\) such that the rate of change of \(V(t)\) can be negative:

\[
\dot{V}(t) = 2(1 - \epsilon)E(t) + 2(1 - \epsilon)\dot{E}(t) + \sigma(t)
\]

\[
= 2(1 - \epsilon)E(t) + [1 + k_1(1 - v^2)] w_x^2 (1, t)
\]

\[
- 2(1 - \epsilon)\dot{E}(t) - \int_0^t x w_x^2 dx - (1 - v^2) \int_0^t x w_x^2 dx
\]

\[
\leq 0, \quad \text{for} \quad t \geq T_2.
\]

(52)

The sum of the first two terms in (52) is positive but smaller than the sum of the remaining terms for large \(T_2\). Combining (43) and (52), we obtain an upper bound for the energy \(E(t)\):

\[
E(t) \leq \frac{V(T_2)}{[2(1 - \epsilon) - C]}, \quad t \geq T_2.
\]

(53)

Hence (53) proves that \(E(t)\) decays as \(\theta/(1/t)\) for large \(t\). q.e.d.

The result (53) describes uniform stabilization of the closed-loop system. The following theorem for the exponential decay of \(E(t)\) can be shown using a semigroup of the closed-loop system.

**Theorem 3:** The energy \(E(t)\) given by (3) decays exponentially to zero along the solutions of (34). That is, there exist constants \(\delta > 0\) and \(M > 0\) such that the following holds.
\begin{align}
E(t) &= Me^{-\omega t}, \quad t \geq 0. \tag{54}
\end{align}

**Proof:** From (37) and (53), the following integral is finite:
\begin{align}
\int_0^\infty E^2(t)dt &= \int_0^{T_1} E^2(t)dt + \int_T^{\infty} E^2(t)dt \\
&\leq \int_0^{T_1} \|S(t)z_0\|^2dt + \int_T^{\infty} \frac{V(t)^2}{(2(1-\epsilon)t-C)^2} dt \\
&< \infty. \tag{55}
\end{align}

Then recall the following semigroup theorem (Pazy, 1983) to obtain the exponential stability of the system:

Let \(S(t)\) be a \(C_0\) semigroup generated by \(-A\) in (34). If for some \(p, 1 < p < \infty\), we have
\begin{align}
\int_0^\infty \|S(t)z_0\|^p dt < \infty, \quad \forall z_0 \in D(A). \tag{56}
\end{align}

Then there exist constants \(K > 0\) and \(\mu > 0\) such that \(\|S(t)\| \leq Ke^{-\mu t}\).

Application of this theorem to (55) gives
\begin{align}
\|z(t)\| \leq \frac{K}{\mu}\|z_0\|e^{-\mu t}. \tag{57}
\end{align}

proving exponential stability of solutions to (34). The energy \(E(t)\) also decays exponentially such that
\begin{align}
E(t) &= \|z(t)\|^2 \leq \frac{K}{\mu}\|z_0\|^2e^{-2\mu t} \\
&= Me^{-\delta t}. \tag{58}
\end{align}

where \(M = \frac{K}{\mu}\|z_0\|^2\) and \(\delta = 2\mu\). q.e.d.

The proofs show that the right boundary force control law (12) with a local velocity feedback stabilizes the vibration energy of the axially moving string exponentially. The exponential stability of the string by the left boundary controls (15) - (17), as well as the other right boundary controls (7) and (9), can be proved similarly.

6 Numerical Comparison

To illustrate the effectiveness and optimality of boundary control, numerical examples are simulated using a finite difference scheme. A mesh of \(N = 400\) nodes along the length of the string and a time step \(\Delta t = 2 \times 10^{-6}\) were used.

Figure 2 shows the reflected wave at the right boundary \(x = 1\) for various control gains of force control with a local velocity feedback. From (22), the amplitude ratio for \(v = 0.5\) is
\begin{align}
\frac{A_l}{A_r} = \frac{M}{3}. \tag{59}
\end{align}

As predicted, the simulation results verify that the amplitude ratio between incident and reflected waves is 1/3 when \(k_f = 0\), and \(A_l/A_r = -1/3\) for \(k_f = 2\) (note that \(|M| = 1\) for the two gains). Also the boundary control set to the optimal gain \(k_f = 1/(1 + v)\) = 0.6667 shows complete dissipation of an incident wave to the controller.

The reflected waves of the string controlled at the upstream boundary \(x = 0\) for various control gains, are shown in Fig. 3. For the boundary control with the optimal gain \(k_f = 1/(1 + v) = 2\), the wave reflection vanishes. From (24), the amplitude \(A_r\) of a reflected wave is greater than that of an incident wave to the controller with a gain \(k_f = -3\), even though the gain is in the stable region \(k_f = -3 < -1/v = -2\). The amplitude ratio \(A_r/A_l = \frac{-3k_f(1 - v) - 1/k_f(1 + v)}{1 - 2.14}\).

Then recall the following semigroup theorem (Pazy, 1983) to obtain the exponential stability of the system:

Let \(S(t)\) be a \(C_0\) semigroup generated by \(-A\) in (34). If for some \(p, 1 < p < \infty\), we have
\begin{align}
\int_0^\infty \|S(t)z_0\|^p dt < \infty, \quad \forall z_0 \in D(A). \tag{56}
\end{align}

Then there exist constants \(K > 0\) and \(\mu > 0\) such that \(\|S(t)\| \leq Ke^{-\mu t}\).

Application of this theorem to (55) gives
\begin{align}
\|z(t)\| \leq \frac{K}{\mu}\|z_0\|e^{-\mu t}. \tag{57}
\end{align}

proving exponential stability of solutions to (34). The energy \(E(t)\) also decays exponentially such that
\begin{align}
E(t) &= \|z(t)\|^2 \leq \frac{K}{\mu}\|z_0\|^2e^{-2\mu t} \\
&= Me^{-\delta t}. \tag{58}
\end{align}

where \(M = \frac{K}{\mu}\|z_0\|^2\) and \(\delta = 2\mu\). q.e.d.

The proofs show that the right boundary force control law (12) with a local velocity feedback stabilizes the vibration energy of the axially moving string exponentially. The exponential stability of the string by the left boundary controls (15) - (17), as well as the other right boundary controls (7) and (9), can be proved similarly.

The energy in the reflected wave depends on the feedback gain \(k_f\) of the right boundary control (Fig. 4). We see that the reflected energy from boundary control is minimum at \(k_f = 1/(1 + v)\) in each case as predicted.

Figures 5 and 6 also illustrate the effectiveness of the optimal boundary control in Section 4.3. The transverse response of an axially moving string controlled by a local velocity feedback \(k_f = 1/(1 + v)\) or \(k_f = 1\) is shown in Fig. 5 for initial conditions \(w(x, 0) = \sin(5\pi x), w_0(x, 0) = 0\) and transport speed \(v = 0.5\). The energy decay for the string controlled by three different controllers is also shown in Fig. 6. The maximum time for the stabilization by left boundary control at \(x = 0\) is the same as that by right boundary control at \(x = 1\), even though a local increase in \(E(t)\) of the left boundary controlled system is shown due to positive energy flux at \(x = 1\). When both boundaries are controlled, the time for the stabilization is \(t \approx T = 7/(1 - v) = 2T\) as predicted in (30).

The transverse response of an initial disturbance around at \(x = 0.5\) by right boundary control with optimal control gain is shown in Fig. 7 for several time steps. The characteristic lines of the initial disturbance in Fig. 8 shows two waves propagating at speed \(1 - v\) and \(1 + v\), respectively. There exist three regions of energy decay. The vibration energy propagating at \(1 + v\) to \(x = 1\) is dissipated under the right boundary control, and negative

![Fig. 4 Dependence of reflected energy on control gain \(k_f\), for various transport speeds \(v = 0.1, 0.3, 0.5, 0.7\) and \(v = 0.9\).](image-url)

![Fig. 5 Response \(w(x, t)\) of an axially moving string with initial conditions \(w(x, 0) = 0.1 \sin(5\pi x)\) and \(w_0(x, 0) = 0\) and transport speed \(v = 0.5\).](image-url)
energy flux produced at \( x = 0 \) causes the decrease of the incoming energy by 66.7 percent: the explicit amount of the energy variation is recently derived by Lee and Mote (1995). Finally the downstream wave reflected from \( x = 0 \) is stabilized at \( x = 1 \). In this example the maximum time for the stabilization is \( t_s = \{0.6/(1 - \nu) + 1/(1 + \nu)\} = 1.8667(T) \), which is the time required for the upstream wave starting at \( x = .6 \) to reflect at \( x = 0 \) and impinge on the right boundary control.

7 Conclusions

1 The stabilization of transverse vibration of an axially moving string by boundary control method without spillover is presented. The boundary control laws obtained by Lyapunov's direct method are implemented through a passive design using a typical viscous damper as well as an active control like a direct velocity feedback control.

2 The time-optimal control for the maximum decay of vibration energy is determined by minimizing the energy of reflected waves at the boundaries. The optimal gain to prevent the complete dissipation of an incident wave, is the ratio of tension to the propagation velocity of an incident wave.

\[
E(t) = \frac{1}{2} \int_0^L w^2(x, t) dx
\]

3 The maximum time required to absorb all vibration energy of the string for any initial disturbance equals the time for a disturbance to propagate the length of the string before reaching the boundary controller. For right or left boundary control with optimal control gains the time is \( t_s = \{1/(1 + \nu) + 1/(1 - \nu)\}(T) \). When both boundaries are controlled, the time for stabilization is \( t_s = 1/(1 - \nu)(T) \).

4 Asymptotic and exponential stability of the string through boundary control are proved.

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References


Fig. 6 Decay of the total energy \( E(t) \) by various boundary control with optimal gain. Initial conditions are \( w(x, 0) = 0.1 \sin (5\pi x) \) and \( w_t(x, 0) = 0 \), and transport speed \( \nu = 0.5 \)

Fig. 7 Characteristic lines and the decay of energy \( E(t) \) for the string translating at \( \nu = 0.5 \) with an initial disturbance around \( x = 0.5 \) by right boundary control with optimal gain


